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The Toda lattice and Kadomtsev–Petviashvili equations

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Abstract. The three-dimensional generalised Toda lattice is introduced and investigated. Its connection to the Kadomtsev-Petviashvili equations is established. We unify the inverse scattering method, Darboux and Bäcklund transformation for the Kadomtsev-Petviashvili, Korteweg-de Vries equations and the Toda lattice.

1. Introduction

Solitary waves were derived as special solutions of the Korteweg-de Vries (KdV) equation when it was formulated in 1985 (see [1]). The stability of these solutions under collisions was investigated by Zabusky and Kruskal [2] 70 years later, using a computer simulation and regarded as a continuum version of the recurrent phenomena originally discovered by Fermi, Pasta and Ulam on a non-linear lattice [3]. Toda intended to find a non-linear discrete equation having these solitary waves as a solution, and found the famous Toda lattice in 1967 [4]. After the introduction of the Toda lattice its importance was quickly and widely recognised by physicists. It appears that the Toda lattice and its different generalisations can describe different physical phenomena and are contained among the soliton equations. These facts have given a strong impetus for a thorough investigation of these equations and recent results in this direction can be found in [5].

Both the Toda lattice and KdV equation have played a very important role in non-linear problems. Some common features of the equations characterised by these solutions, i.e. the soliton equation, became known by 1970, such as the inverse method, the Bäcklund transformation method and also infinitely many conserved currents. In mathematical studies of non-linear problems, it is very interesting to understand various equations from a common basis. Recently Saitoh [6] proposed a method for unifying the KdV equation and the Toda lattice by introducing a generalised Toda lattice. This method clarifies the relation between a soliton equation of continous variables and its discrete version. In this approach the KdV equation is derived from the Toda lattice through a transformation characterised by a parameter in such a way that the integrability is preserved for all values of the parameter.

On the other side the KdV equation can also be recovered from the Kadomtsev-Petviashvili (KP) [7] equation. The KP equation is the non-linear partial differential equation in three independent variables t, x, y while KdV is in two: t, x. The KdV equation is obtained by neglecting the dependencies in the y variable in the KP equation. Both equations can be used to describe the evolution of relatively long water waves of moderate amplitudes as they propagate in shallow water. Both equations are completely integrable. Recently it appeared that the KP equation played an important role in string theory [8].

For these reasons it is reasonable to ask: is it possible to unify these equations i.e. KP, KdV and Toda lattice, simultaneously? In this paper we solve this problem. To this aim we introduce a more general Toda lattice than proposed by Saitoh. Our method utilises the Saitoh transformation characterised by a parameter and by 'addition' of one more dimension similar to the KP equation. As the result we obtain the three-dimensional Toda lattice. In the sense explained below the KP equation is given by the limit of our generalised Toda lattice which is a transformed version of the Toda lattice itself. Then the KdV equation is obtained from the KP equation by standard procedures. Therefore the KP and KdV systems can be said to be in the framework of the Toda lattice. The three-dimensional Toda lattice just defined extends our knowledge of the still small class of integrable equations in 2+1 dimensions.

This paper is organised as follows. In § 2 we recapitulate the construction of Saitoh in which she defined a generalised two-dimensional Toda lattice. In § 3 we introduce our construction. The next three sections contain descriptions of the constructions of the inverse scattering method, Darboux and Bäcklund transformations for our model, respectively. In all these sections we investigate the transformation characterised by a parameter in such a way as to obtain the connection between the continuous and discrete version. The last section contains concluding remarks.

2. The generalised two-dimensional Toda lattice

The system of equations obtained by Toda can be written down as follows:

$$\frac{\partial^2}{\partial\tau^2}\ln(1+f_n)=f_{n+1}+f_{n-1}-2f_n \tag{2.1}$$

where n denotes the lattice site and f_n the force between particles. Saitoh introduced the following transformation with the parameter $0<h\leqslant 1$:

$$\tau=t/h^3 \tag{2.2}$$

$$n=x/h+(1/h^2-h^2)t/h \tag{2.3}$$

$$f_n=h^2u_n(\tau)=h^2u(x,t) \tag{2.4}$$

in which the variable x which specifies the site of a lattice is no longer discrete but is now continuous. Applying the transformation (2.2)-(2.4) to (2.1) we obtain the generalised Toda lattice:

$$\left[\frac{\partial}{\partial t}-\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}\right]^2\ln(1+h^2u(x,t))=\frac{1}{h^4}[u(x+h,t)+u(x-h,t)-2u(x,t)]. \tag{2.5}$$

If we put $h=1$ in (2.5) we obtain the Toda lattice while for $h\to 0$ (2.5) reduces to the KdV equation

$$\frac{\partial}{\partial t}u+\frac{1}{4}\frac{\partial}{\partial x}u^2+\frac{1}{24}\frac{\partial^3}{\partial x^3}u=0. \tag{2.6}$$

In that sense the KdV equation is given as the limit of the generalised Toda lattice. Notice that the right-hand side of (2.5) can be considered as the function which depends on the three different points $x\pm h$ and x. For that reason, such a generalisation sometimes is called non-local [9] but in the next section we will not use this name.

3. The generalised three-dimensional Toda lattice

Let us discuss why (2.5) can be called the two-dimensional Toda lattice. On the one hand it can be named in such a way because this equation describes the behaviour of the function which depends on the two variables. On the other hand notice that the left-hand side of (2.5) is not exactly a 'two dimensional' generalisation of the d'Alembertian or Laplacian operator in two-dimensional space. Indeed the operator on the left-hand side of (2.5) is a second power of the same operator.

In order to avoid the misunderstanding in this interpretation and with the two-dimensional Toda lattice proposed by Mikhajlov [10] and Fordy and Gibbons [11] which can be written as

$$\frac{\partial^2}{\partial\tau\partial\tau'}\ln(1+f_n)=f_{n+1}+f_{n-1}-2f_n \tag{3.1}$$

where $f_n=f_n(\tau,\tau')$ and n denotes the lattice site, we will use the name of two-dimensional Toda lattice while we call (2.5) the generalised two-dimensional Toda lattice.

Therefore from this point of view one can ask: is it possible to apply the two-dimensional d'Alembertian or Laplacian operator with the Saitoh trick, i.e. with $1/h^2-h^2$, simultaneously to the left-hand side of (3.1)? To see that it *is* possible we shall introduce the following transformation with the parameter $0<h\leqslant 1$:

$$\tau=t/h^2+\frac{h}{2\sigma}y \qquad \tau'=t/h^3-\frac{h}{2\sigma}y \tag{3.2}$$

$$n=x/h\pm\left(\frac{1}{h^2}-h^2\right)t/h \tag{3.3}$$

$$f_n=h^2\tilde{u}_n(\tau,\tau')=h^2u(x,y,t) \tag{3.4}$$

where now x is no longer discrete but continuous and t and y are two independent variables and $\sigma^2=\pm1$. Applying the transformation (3.2)-(3.4) to (3.1) we obtain

$$\begin{aligned}&\left\{\left[\frac{\partial}{\partial t}\mp\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}\right]^2-\frac{\sigma^2}{h^2}\frac{\partial^2}{\partial y^2}\right\}\ln(1+h^2u(x,y,t))\\ &\quad=\left[\frac{\partial}{\partial t}\mp\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial y}+\frac{\sigma}{h}\frac{\partial}{\partial y}\right]\left[\frac{\partial}{\partial t}\mp\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}-\frac{\sigma}{h}\frac{\partial}{\partial y}\right]\ln(1+h^2u)\\ &\quad=\frac{1}{h^4}[u(x+h,y,t)+u(x-h,y,t)-2u(x,y,t)].\end{aligned} \tag{3.5}$$

Equation (3.5) is called the three-dimensional generalised Toda lattice in the following. For $h=1$ our equation reduces to the two-dimensional Toda lattice (3.1) while for $h\to0$ it is easy to see that we then obtain

$$\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial t}u+\tfrac{1}{2}uu_x+\frac{1}{24}\frac{\partial^3}{\partial x^3}u\right)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}u=0 \tag{3.6}$$

the KP equation. More exactly, the KP I if $\sigma^2=1$ and KP II if $\sigma^2=-1$. The KdV equation is recovered from the KP equation if we neglect the y dependence in (3.6) and put the constant of integration equal to zero.

In this sense the KP and KdV equations are given by the limit of our three-dimensional generalised Toda lattice which is a transformed version of the Toda lattice itself.

4. The inverse scattering method for the three-dimensional generalised Toda lattice

The inverse scattering transformation (IST) is one of the most powerful methods of solving the soliton equations. First this method was used for the KdV equation and later it has been extended by Zakharov and Schabat [12] and Ablowitz *et al* [13] to other two-dimensional soliton equations. From the conceptual point of view the steps associated with the implementation of IST are now quite clear. The heart of this method is the two-dimensional Schrödinger equation in which the potential is the solution of the given non-linear equation. In order to reconstruct this potential one can use the inverse method. Generally speaking, the class of solvable equations for which IST can be initiated results from the compatibility of two operators, which we shall call L and M:

$$L\psi = \lambda\psi \tag{4.1}$$

$$\psi_t = M\psi. \tag{4.2}$$

The associated non-linear evolution equation is obtained via compatibility, given by

$$L_t + LM - ML = 0 \tag{4.3}$$

if and only if $\lambda_t = 0$. Equations (4.1) and (4.2) constitute Lax's famous result. L and M may be matrix valued operators of arbitrary order. The operator L may be differential, integro-differential, discrete or even a purely linear algebraic system [14].

For the KP equations this inverse scattering method was first applied by Manakov [15]. Here the situation is conceptually similar to the two-dimensional case. The KP equations are obtained via the compatibility condition on L and M operators but the Schrödinger equation becomes time dependent.

The inverse scattering method is also used on the Toda lattice. Flaschka [16] and Manakov [17] successfully used this method on the one-dimensional Toda lattice. Fordy and Gibbons [11] (see also Mikhajlov [10]) discovered IST for a two-dimensional periodic Toda lattice by the factorisation of the Nth-order elliptic scalar differential operator. This result can also be easily extended onto the infinite two-dimensional Toda lattice.

We now show that these methods follow from a common basis from the IST for our three-dimensional generalised Toda lattice. First let us define this method for our model. Notice that, due to the factorisation of the operator in (3.5), one can easily write down the inverse scattering method, which is

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}+\frac{\sigma}{h}\frac{\partial}{\partial y}\right]\psi(n) &= -\frac{1}{h^3}\exp\left(\frac{r(n)}{2}\right)\psi(n-1)-\frac{1}{2}\left[\frac{\partial}{\partial t}+\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}+\frac{\sigma}{h}\frac{\partial}{\partial y}\right]\\ &\quad\times\left(\sum_{i=-\infty}^{n} r(i)\right)\psi(n)\end{aligned} \tag{4.4}$$

$$\left[\frac{\partial}{\partial t}+\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}-\frac{\sigma}{h}\frac{\partial}{\partial y}\right]\psi(n)$$
$$=\frac{1}{h^3}\exp\left(\frac{r(n+1)}{2}\right)\psi(n+1)+\frac{1}{2}\left[\frac{\partial}{\partial t}+\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}-\frac{\sigma}{h}\frac{\partial}{\partial y}\right]$$
$$\times\left(\sum_{i=-\infty}^{n} r(i)\right)\psi(n) \tag{4.5}$$

where $\psi(n)=\psi(x,y,t)$, $\psi(n\pm1)=\psi(x\pm h,y,t)$ and

$$h^2u(x,y,t)=\exp[\varphi(n,y,t)-\varphi(n-1,y,t)]-1=\exp[r(n)]-1 \tag{4.6}$$
$$\varphi(n,y,t)=\varphi(x,y,t) \qquad \varphi(n\pm1,y,t)=\varphi(x\pm h,y,t) \tag{4.7}$$
$$\varphi(n,y,t)=\sum_{i=-\infty}^{n} r(i). \tag{4.8}$$

Notice that the original equation (3.5) with the '+' sign is obtained as a compatibility condition on (4.4) and (4.5). We choose the '+' sign for simplicity. However all calculus can be carried out for the '−' sign also.

Our IST is equivalent for $h=1$ with IST for the two-dimensional Toda lattice discovered by Mikhajlov [10] or with IST considered by Fordy and Gibbons [11] after the transformation

$$\psi(n)\to\exp\left(\frac{\varphi(n)}{2}\right)\psi(n). \tag{4.9}$$

Now we show that our IST for $h\to0$ reduces to the IST for KP I or KP II. To see it let us rewrite (4.4) and (4.5) as

$$2\left[\frac{\partial}{\partial t}+\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}\right]\psi(n)$$
$$=\frac{1}{h^3}\left[\exp\left(\frac{r(n+1)}{2}\right)\psi(n+1)-\exp\left(\frac{r(n)}{2}\right)\psi(n-1)\right]$$
$$-\frac{\sigma}{h}\left(\frac{\partial}{\partial y}\sum_{i=-\infty}^{n} r(i)\right)\psi(n) \tag{4.10}$$

$$2\frac{\sigma}{h}\frac{\partial}{\partial y}\psi(n)=-\frac{1}{h^3}\left[\exp\left(\frac{r(n)}{2}\right)\psi(n-1)+\exp\left(\frac{r(n+1)}{2}\right)\psi(n+1)\right]$$
$$-\left[\frac{\partial}{\partial t}+\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}\right]\left(\sum_{i=-\infty}^{n} r(i)\right)\psi(n). \tag{4.11}$$

To investigate the limit $h\to0$ let us use the following approximation formulae:

$$h^2\omega(n,y,t)=h^2\omega(x,y,t)=r(n)=h^2\omega(x)=h^2\omega \tag{4.12}$$
$$\lim_{h\to0}\sum_{i=-\infty}^{n} r(i)=h\int_{-\infty}^{x}u(x')\,dx'=\varphi(n)=\varphi(x) \tag{4.13}$$
$$\lim_{h\to0}\exp\left(\frac{r(n+1)}{2}\right)\psi(n+1)=\left[\psi+h\psi_x+\tfrac{1}{2}h^2\psi_{xx}+\frac{h^3}{3!}\psi_{xxx}+\tfrac{1}{2}h^2\omega\psi+\tfrac{1}{2}h^3\omega\psi_{x}+\tfrac{1}{4}h^3\omega_x\psi\right] \tag{4.14}$$
$$\lim_{h\to0}\exp\left(\frac{r(n)}{2}\right)\psi(n-1)=\left[\psi-h\psi_x+\tfrac{1}{2}h^2\psi_{xx}-\frac{h^3}{3!}\psi_{xxx}+\tfrac{1}{2}h^2\omega\psi-\tfrac{1}{2}h^3\omega\psi_x-\tfrac{1}{4}h^3\omega_x\psi\right] \tag{4.15}$$

where

$$\psi = \psi(n) = \psi(x) \qquad \psi_x = \frac{\partial}{\partial x}\psi. \tag{4.16}$$

Substituting (4.12)-(4.16) into (4.10) and (4.11) and equating the terms in the same power of h we obtain after scaling the function

$$\psi \to \exp\left[-\left(\frac{y}{\sigma h^2}\right)\right]\psi \tag{4.17}$$

$$\psi_t = \frac{1}{6}\psi_{xxx} + \tfrac{1}{2}\omega\psi_x + \tfrac{1}{2}\omega_x\psi - \tfrac{1}{2}\sigma\int^x \omega_y\psi\,\mathrm{d}x' \tag{4.18}$$

$$\sigma\psi_y = -\tfrac{1}{2}\psi_{xx} + 2\omega\psi. \tag{4.19}$$

Equations (4.18) and (4.19) are the IST for KP I or KP II discovered and discussed by Manakov [15]. Now the IST for KdV can be easily recovered from (4.18) and (4.19) by neglecting the y dependence in (4.18) and putting $\sigma\psi_y = \lambda\psi$ in (4.19). In that way we unify IST from KP and KdV equations with IST for the three-dimensional generalised Toda lattice. In that sense KP and KdV are transformed versions of our Toda lattice itself.

5. The Darboux transformation

The Darboux transformation can be applied for the explicit integration of linear evolution equations with the scalar or matrix valued coefficients. Probably Matveev was the first who used this concept for the integration of the Toda lattice [18]. In order to find this transformation for our model one can easily prove that when $\varphi(n)$ with (4.6) is the solution of equation (3.5) then

$$\varphi'(n) = \tfrac{1}{2}(\varphi(n) + \varphi(n-1)) - \ln(\psi(n-1)/\psi(n)) \tag{5.1}$$

also satisfy (3.5) if $\psi(n)$ satisfy (4.4) and (4.5). This is the desired Darboux transformation for our Toda lattice.

In the continuum limit $h \to 0$ using the formula (4.13) we obtain

$$h\int^x \omega' = h\int^x \omega - \tfrac{1}{2}h^2\int^x \omega_x - \ln(l - h\psi_x/\psi). \tag{5.2}$$

Using the Taylor expansion of the ln function and keeping the terms in the first power of h, we obtain

$$\int^x \omega' = \int^x \omega + \tfrac{1}{4}\psi_x \tag{5.3}$$

or in the equivalent form as

$$\omega' = \omega + \partial^2_{xx}\ln\psi. \tag{5.4}$$

Equations (5.3) and (5.4) define the Darboux transformation for the KP I or KP II equations discovered by Matveev [19]. Interestingly the same formula holds for the KdV equation. The verification of this statement is straightforward. It is enough to substitute (5.3) or (5.4) into (3.6) and use the fact that ψ satisfies (4.18) and (4.19). Now one can use this Darboux transformation to get the analogue of the soliton solutions of our Toda lattice and then to investigate the limiting procedure to obtain the soliton solutions of the KP equations. In this sense the Darboux transformation of our three-dimensional Toda lattice contains the Darboux transformation of the KP as well as KdV equations.

6. The Bäcklund transformation

Derivation of the Bäcklund transformation (BT) for any completely integrable non-linear equations is usually considered to be the ultimate goal. The knowledge of this transformation allows us to construct the huge class of solutions to the given equation. In one point of view the BT is obtained via a gauge transformation of the corresponding Lax operators and this gauge theoretical point of view has been pursued by numerous researchers [20]. It is the analogue of the Darboux transformation in the theory of the Schrödinger equation and the BT for the KdV equation.

In this section we would like to study the interconnection between different BT, i.e. KP, KdV and Toda lattice, and unify these three Bäcklund transformations. We will not discuss here how one can obtain soliton solutions by this method.

First let us notice that Toda [21] derived BT for the KdV equation by continuous approximation from the BT of the one-dimensional Toda lattice. In his derivation, one has to be very careful to find the centre of Taylor expansion in order to perform the approximation procedure. In the Saitoh approach the situation become much easier because the choice of the centre becomes rather natural. The same situation appears in our case. Indeed the BT for the three-dimensional Toda lattice has the form

$$\left[\frac{\partial}{\partial t}-\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}-\frac{\sigma}{n}\frac{\partial}{\partial y}\right](\varphi(n+1)-\tilde{\varphi}(n))$$
$$=\frac{1}{h^3}\{\exp[\tilde{\varphi}(n+1)-\varphi(n+1)]-\exp[\tilde{\varphi}(n)-\varphi(n)]\} \tag{6.1}$$

$$\left[\frac{\partial}{\partial t}-\left(\frac{1}{h^2}-h^2\right)\frac{\partial}{\partial x}+\frac{\sigma}{h}\frac{\partial}{\partial y}\right](\tilde{\varphi}(n)-\varphi(n))$$
$$=\frac{1}{h^3}\{\exp[\varphi(n+1)-\tilde{\varphi}(n)]-\exp[\varphi(n)-\tilde{\varphi}(n-1)]\}. \tag{6.2}$$

Here $\varphi(n)$ and $\tilde{\varphi}(n)$ are the old and new solutions of (3.5) respectively. For $h=1$ our formulae reduce to the Bäcklund transformation for the two-dimensional Toda lattice considered by Fordy and Gibbons [11]. In order to find the limit $h\to 0$ in (6.1) and (6.2) let us use two types of expansions of $\varphi(n)$. For (6.1) let us expand $\varphi(n)$ and $\tilde{\varphi}(n)$ in the MacLaurin series around $x+h/2$. For example we obtain

$$\varphi(n+\tfrac{1}{2}+\tfrac{1}{2})-\tilde{\varphi}(n+\tfrac{1}{2}-\tfrac{1}{2})=h\int^{x+h/2}[\omega(x')-\tilde{\omega}(x')]\,\mathrm{d}x'+\tfrac{1}{2}h^2(\omega+\tilde{\omega})+\tfrac{1}{8}h^3(\omega-\tilde{\omega})_x. \tag{6.3}$$

We expand $\varphi(n)$ and $\tilde{\varphi}(n)$ in (6.2) around x. In the following we assume that both ω and $\tilde{\omega}$ are analytic functions of x in order to compare the expansions of (6.1) and (6.2). Using this type of expansion for (6.1) and (6.2), respectively, and equating the coefficients in the same power of h up to first order we obtain

$$\sigma\frac{\partial}{\partial y}\int^x(\tilde{\omega}-\omega)\,\mathrm{d}x'=\left(\frac{\omega+\tilde{\omega}}{2}\right)_x-(\omega-\tilde{\omega})\int^x(\omega-\tilde{\omega})\,\mathrm{d}x' \tag{6.4}$$

$$\frac{\partial}{\partial t}\int^{x}(\omega-\tilde{\omega})\,dx'-\frac{\sigma}{2}\frac{\partial}{\partial y}(\omega+\tilde{\omega})=\tfrac{1}{12}(\omega-\tilde{\omega})_{xx}-\tfrac{1}{2}(\omega-\tilde{\omega})\left(\int^{x}(\omega-\tilde{\omega})\,dx'\right)^{2} \tag{6.5}$$

$$\frac{\partial}{\partial t}\int^{x}(\omega-\tilde{\omega})\,dx'=\tfrac{1}{6}(\omega-\tilde{\omega})_{xx}+\tfrac{1}{2}\left[\omega^{2}-\tilde{\omega}^{2}+\left(\int^{x}(\omega-\tilde{\omega})\,dx'\right)(\omega-\tilde{\omega})_{x}\right]$$
$$+\tfrac{1}{2}(\omega-\tilde{\omega})\left(\int^{x}(\omega-\tilde{\omega})dx'\right) \tag{6.6}$$

formulae (6.4)-(6.6) can be reduced to the simpler form if we use the following transformation:

$$\omega=-\frac{d}{dx}q \qquad \tilde{\omega}=-\frac{d}{dx}\tilde{q} \tag{6.7}$$

then (3.6) transforms to

$$\frac{\partial}{\partial t}q-\tfrac{1}{4}q_{x}q_{x}+\tfrac{1}{24}q_{xxx}+\tfrac{1}{2}\sigma^{2}\int^{x}q_{yy}\,dx'=0. \tag{6.8}$$

Now if we subtract (6.6) from (6.5) we obtain

$$\frac{\partial}{\partial t}\int_{-x}^{x}(\tilde{q}-q)\,dx'+\tfrac{1}{4}\sigma\frac{\partial}{\partial y}(q+\tilde{q})=\tfrac{1}{24}(q-\tilde{q})_{xx}+\tfrac{1}{6}(q-\tilde{q})^{3}-\tfrac{1}{4}(q-\tilde{q})(q+\tilde{q})_{x} \tag{6.9}$$

and formula (6.4) becomes

$$2\sigma\frac{\partial}{\partial y}\int^{x}(\tilde{q}-q)\,dx'=(q+\tilde{q})_{x}-(q-\tilde{q})^{2}. \tag{6.10}$$

In this way (6.9) and (6.10) are the Bäcklund transformations for the KP I or KP II equations first discovered by Chen [22] and later disucssed by many authors [23].

In this sense we unified the Bäcklund transformation of the KP equations with the Bäcklund transformation for our three-dimensional generalised Toda lattice.

In order to obtain the BT for KdV from (6.9) and (6.10) it is enough to assume

$$\frac{\partial}{\partial y}(q+\tilde{q})=0 \tag{6.11}$$

$$\sigma\int^{x}(\tilde{q}-q)_{x}\,dx'=\text{constant} \tag{6.12}$$

and differentiate (6.9) with respect to x. Then our formulae (6.8)-(6.10) reduce to the BT for KdV obtained by Wadati *et al* [24].

7. Concluding remarks

In this paper we have shown that the KP equations, KdV and Toda lattice can be unified by introducing the concept of the three-dimensional generalised Toda lattice. Then we investigated how the inverse scattering method, Darboux and Bäcklund transformations are connected among themselves. There are quite a number of other remarkable properties, which are shared by most of the intergrable equations and which are believed to indicate solvability such as: a bi-Hamiltonian formulation and the existence of a hereditary recursion operator. In the case of the KP equations the non-existence statements of the bi-Hamiltonian structure were formulated [25, 26]. However Fokas and Santini recently showed [27] that the bi-Hamiltonian structure of KdV and KP equations follow from the same operator structure. In their construction one requires the essential use of a formalism for generalised functions (distributions) with bilocal arguments. It will be very interesting to use our approach in the investigation of the bi-Hamiltonian structure of the KP equation, because the bi-Hamiltonian structure of the Toda lattice had been established earlier by Leo *et al* [28], and then compare this with the formalism proposed by Fokas and Santini. The paper in this direction is in preparation.

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References

[1] Bullough R and Caudrey P (ed) 1980 *Solitons* (Berlin: Springer)
[2] Zabusky N and Kruskal M 1965 *Phys. Rev. Lett.* **15** 240
[3] Fermi E, Pasta J and Ulam S 1965 *Collected Papers of E Fermi* vol 2 (Chicago: University of Chicago Press) 1978
[4] Toda M 1967 *J. Phys. Soc. Japan* **22** 431
[5] Popowicz Z 1988 *Solitons Introduction and Applications* ed M Lakshmanan (Berlin: Springer) p 212
[6] Saitoh N 1985 *J. Phys. Soc. Japan* **154** 3261
[7] Kadomtsev B and Petviashvili V 1970 *Sov. Phys.-Dokl.* **15** 539
[8] Saito S 1987 *Phys. Rev. Lett.* **59** 1798
[9] Salle M 1982 *Theor. Math. Phys.* **53** 227 (in Russian)
[10] Mikhajlov A 1979 *JETP Lett.* **30** 443
[11] Fordy A and Gibbons J 1980 *Commun. Math. Phys.* **77** 21
[12] Zakharov V and Schabat A 1971 *Sov. Phys.-JETP* **61** 118
[13] Ablowitz M, Kaup D, Newell A and Segur H 1973 *Phys. Rev. Lett.* **30** 1262
[14] Ablowitz M and Fokas A 1982 *Nonlinear Phenomena* (*Lecture Notes in Phys.* **189**) (Berlin: Springer)
[15] Manakov S 1981 *Physics* D **3** 420
[16] Flaschka H 1974 *Prog. Theor. Phys.* **51** 703
[17] Manakov S 1974 *Sov. Phys.-JETP* **67** 543
[18] Matveev V 1979 *Lett. Math. Phys.* **3** 217
[19] Matveev V 1979 *Lett. Math. Phys.* **3** 213
[20] Levi D, Ragnisco O and Sym A 1984 *Nuovo Cimento* **83B** 134
[21] Toda M 1976 *Prog. Theor. Phys. Suppl.* **59** 1
[22] Chen H 1975 *J. Math. Phys.* **16** 2382
[23] David D, Levi D and Winternitz P 1986 *Phys. Lett.* **118A** 390
[24] Wadati M, Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 119
[25] Fokas A and Fuchssteiner B 1981 *Phys. Lett.* **86A** 341
[26] Zakharvo V and Konopelchenko B 1984 *Commun. Math. Phys.* **94** 483
[27] Fokas A and Santini P 1987 Bi-Hamiltonian formualtion of the Kadomtsev-Petviashvili and Benjamin-Ono equation *Preprint* INS 79, Clarkson University
[28] Leo M, Leo R, Solani G and Solombrino L 1984 *Lett. Math. Phys.* **8** 267